Self-Referential Grids

by Jack Cats

math@JackCats.com
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1 Introduction

The inspiration for this comes from Youtube channel "Cracking The Cryptic" and their video on a particular sudoku-like puzzle, which you can find here. The title of the puzzle in question is "Self-determination", by Stephane Bura, and although watching/solving is not necessary to understand this document, it is recommended to help get a sense of the ideas involved. One might note that the puzzle uses 1-indexing, whereas this document uses 0-indexing; the choice of index corresponds to a relabelling, and does not change the quantity or quality of any solutions.

A famous result in the puzzle domain is that the minimum number of clues needed to uniquely determine a sudoku solution is 17. Although Self-determination is an 8x8 grid, the fact that only 1 clue is required indicates that the rules somehow are quite restrictive, while still allowing for any solutions at all. A logical next question might be to determine exactly how many 8x8 grids can be filled with the rules as stated in the puzzle: one would find the answer to be 2.

This document provides some minor proofs about patterns found in the generalized Self-determination puzzle.

2 Definitions

We start with a grid G of size $n \times n$, where n is a positive integer, filled with cells $c : c \in \mathbb{Z}_n$ (the set of integers modulo n). Let $g(r, c)$ be the function that returns the value of the cell at row r and column c in G. We say a pair of cells $A = g(a_r, a_c)$ and $B = g(b_r, b_c)$ is Referential iff $g(A, B) = a_r$; we'll also say A is the *row-index* of the pair, and B is the *column-index*. A Horizontal Domino is a pair of cells $A = g(a_r, a_c)$ and $B = g(a_r, a_c + 1)$.

We define a Self Referential grid to be a grid G that satisfies the following conditions:

CN1a. Each row r in G contains each element of $\mathbb{Z}_n$ exactly once.

CN1b. Each column c in G contains each element of $\mathbb{Z}_n$ exactly once.

CN2. Each Horizontal Domino in G with $a_c < n - 2$ is Referential.

We will also define some notation for which the motivation will become clear later on. Let the symbols $\dot{+}$ and $\dot{-}$ represent binary operators for members of $\mathbb{Z}_n$, defined as follows:

$$a \dot{+}_n b = \begin{cases} a + b \bmod n-1 & a < n-1 \text{ and } b < n-1 \\ n-1 & a = n-1 \text{ or } b = n-1 \end{cases}$$

$$a \dot{-}_n b = \begin{cases} a - b \bmod n-1 & a < n-1 \text{ and } b < n-1 \\ n-1 & a = n-1 \text{ or } b = n-1 \end{cases}$$

The subscript n will usually be omitted, assumable from context. In some places, the notation $\dot{a}$ will refer to the value of $a \dotplus_n 1$.

Lastly, we say B_n is the set of all grids G of size $n \times n$ that are Self Referential, and $A(n) = |B_n|$ (OEIS consistency).

3 Discoveries

In identifying solutions for grids of different sizes, we can note a common observation. For $a < n - 2$, $g(a, a+1) = n - 1$. A proof by contradiction is as follows:

1. Assume G is Self-referential, and $g(a, a+1) \neq n - 1$.
2. Let $p = g(a, a)$ and $q = g(a, a+1) \neq n - 1$.
3. Then $g(p, q) = a$ by CN2.
4. Since $q \neq n - 1$, we will analyze two cases: $q = 0$ and $q \neq 0$.
5. If $q = 0$, then (p, q) is the row-index of some Referential Horizontal Domino. Such a domino describes the location of p in row a.
6. We know by CN1a that p is in row a exactly once, so the domino describes $g(a, a) = p$.
7. The column-index corresponding to (p, q) is in position $(p, q+1)$, AKA the same row.
8. But the column-index must be a, so $g(p, q+1) = a$.
9. This contradicts CN1a, so $q \neq 0$.
10. If $q \neq 0$, then (p, q) is the column-index of some Referential Horizontal Domino. Such a domino describes the location of p in column a.
11. We know by CN1b that p is in column a exactly once, so the domino describes $g(a, a) = p$.
12. The row-index corresponding to (p, q) is in position $(p, q-1)$, AKA the same row.
13. But the row-index must be a, so $g(p, q-1) = a$.
14. This contradicts CN1b, so $g(a, a+1) = n-1$, a contradiction of the initial assumption.
15. Therefore, $g(a, a+1) = n - 1$. (Result 1)

From this, we note that for any grid of size $n \times n$, we can place $(n-2)$ $(n-1)$s. Now, there are 2 $(n-1)$s missing, in two possible cases:

1. $g(n-1, n-1) = n-1$ and $g(n-2, 0) = n-1$.
2. $g(n-1, 0) = n-1$ and $g(n-2, n-1) = n-1$.

Case 1 can be proven to be the only possible case by contradiction:

1. $g(g(n-1, 0), g(n-1, 1)) = n-1$
2. Assuming case 2, $g(n-1, g(n-1, 1)) = n-1$
3. Thus, $g(n-1, 1) = 0$
4. From Result 1, $g(0, 1) = n-1$
5. $g(g(n-1, 1), g(n-1, 2)) = n-1$
6. $g(0, g(n-1, 2)) = n-1$
7. Thus, $g(n-1, 2) = 1$
8. $g(g(0, 1), g(0, 2)) = 0$
9. $g(n-1, g(0, 2)) = 0$
10. $g(0, 2) = 1$
11. Statement 7 and 10 put the value 1 twice in column 2, a contradiction.
12. Therefore, case 1 is always correct. (Result 2)

Results 1 & 2 can be combined into a simpler statement using our notation.

S0. $g(a, \dot{a}) = n-1$

We'll demonstrate another result in plain language. Since the value of a cell at $(a, a+1)$ is $n-1$, then a in row $n-1$ must be immediately followed by $a+1$, except when $a = n-2$, in which case a is followed by 0. We form this as the statement:

S1. For $a, b \in \mathbb{Z}_n$, if $g(n-1, a) = b$, then $g(n-1, \dot{a}) = \dot{b}$

Note that the case $a = n-1$ implies $b = n-1$ via S0, so S1 is consistent with the definition $(n-1) \dot{+} 1 = n-1$. Another result which builds from S0 and S1 is:

S2. For $a, b \in \mathbb{Z}_n$, if $g(a, \ddot{a}) = b$, then $g(\dot{a}, \dot{\ddot{a}}) = \dot{b}$

We will now prove another interesting result.

1. Let $a, r : a < n-2$ and $r = g(a, a)$
2. Then (a, a) and $(a, a+1)$ form a Referential Horizontal Domino
3. By S0, $g(a, a+1) = n-1$

4. Let $r_i = g(r, i)$ for $i \in \mathbb{Z}_n$.

5. By the previous two statements, $g(g(a, a), g(a, a+1)) = a = g(r, n-1)$

6. Thus, $r_{n-1} = a$

7. Then, the location of r in row r_c is known for $c \neq n-2$

8. Similarly, the location of r in column r_c is known for $c \neq 0$

9. Thus, $g(r_{n-2}, r_0) = r$

10. Therefore the pair $(r, n-2), (r, 0)$ is Referential (Result 3)

The pair $(k, n-1), (k, n-1)$ can trivially be shown to be Referential. The combination of this, S1, Result 3, the original constraint CN2, and the Pidgeonhole Principle (for Result 3 where $a = n-2$) can prove this modified version of CN2 to be true for all Self-referential grids:

CN2'. For $r, c \in \mathbb{Z}_n$, the pair $(r, c), (r, \dot{c})$ is Referential.

And now some other minor results which we will not prove here:

S3. Each cell is part of two CN2' pairs, one as the row-index and one as the column-index.

S4. Each CN2' pair is unique, and thus Refers to one pair and is Referred to by one pair.

S5. The cell at (r, c) is the row-index of the pair which is the row-index of (r, c)

4 Conjectures

Using these discoveries, we can improve the efficiency of computer algorithms designed to find all possible solutions to the generalized Self-determination puzzle. In doing so, we note some further patterns in solutions, for which a proof is not yet known. A brute-force search for solutions up to $n = 16$ has been conducted, all solutions satisfying the following conjectures:

Q0. For $r, c, a \in \mathbb{Z}_n$, if $g(r, c) = a$, then $g(\dot{r}, \dot{c}) = \dot{a}$

Q1. If $A(n) \neq 0$, then n is a power of a prime.

Q1a. $n = 3$ and $n = 7$ are the only prime powers for which $A(n) = 0$.

Note that S0, S1, and S2 (proven) are specific cases of Q0.

Assuming Q0, the search space is significantly reduced. If a single row is filled, then the rest of the grid is determined.

5 Further Analysis

For the rest of this analysis, a grid for which Q0 holds will be referred to as a Sequential grid (since the diagonals are filled in a sequence), and we will introduce a new constrant CN3, that a grid must be Sequential. Also, let $\hat{A}(n) = |\hat{B}_n|$ where $\hat{B}_n$ is the set of all Sequential Self-referential grids. If the problem before was attempting to find all Self-referential grids, the remainder of this analysis will be on all Sequential Self-referential grids. Q0, then, is equivalent to the statement that all Self-referential grids are Sequential, and would demonstrate CN3 to be redundant.

Adapting the computer search method to find only all Sequential grids, we find $\hat{A}$ starts the same as the previous sequence, but can be extended further much more easily. For $1 \leq n \leq 32$:

$$\hat{A} = [1, 2, 0, 1, 2, 0, 0, 2, 2, 0, 4, 0, 4, 0, 0, 2, 8, 0, 8, 0, 0, 0, 8, 0, 4, 0, 4, 0, 12, 0, 8, 6]$$

Q0 splits any solution into a set of Sequential diagonal bands, along with the last column and last row being Sequential. Illustrated with an example grid of size 8, with one diagonal emphasized:

G	0	1	2	3	4	5	6	7
0	1	7	2	**5**	0	4	3	6
1	4	2	7	3	**6**	1	5	0
2	6	5	3	7	4	**0**	2	1
3	3	0	6	4	7	5	**1**	2
4	**2**	4	1	0	5	7	6	3
5	0	**3**	5	2	1	6	7	4
6	7	1	**4**	6	3	2	0	5
7	5	6	0	1	2	3	4	7

We'll refer to diagonals by the row where they intersect column 0, so the highlighted diagonal is diagonal $D = \mathbf{4}$. The diagonal any cell (r,c) is in can be calculated as $D = r \dot{-} c$ (except for cells in the last column).

When taking the difference between any two cells in the same column (or row) and adjacent diagonals, we see the difference is constant for those diagonals. It can then be shown that the difference $d_r = g(r+1, 0) - g(r, 0)$ must be unique for $0 \leq r \leq n-3$. In fact, this list of differences must always obey certain conditions to form a valid grid:

SQ0. $2 \leq d_i \leq n-2$

SQ1. Each difference is unique, otherwise two equal differences would result in the same Referential Domino in the same row, or different RDs not in the same row but pointing to the same other cell.

SQ2. Differences at opposite ends of the list add to n, due to symmetry.

SQ3. Since the differences "point" to a different diagonal in the grid, the diagonal it points to must "point back".

SQ4. The sum of k consecutive differences must not be congruent to k mod $n-1$, or else this would result in two cells in the same row (or column) being equal.

SQ5. For each difference d_i, and the difference it points to $d_{i'}$, the sum of the differences prior to d_i plus the sum of the differences prior to $d_{i'}$, mod $n-1$ must be the same for all i, and this number cannot be odd if n is odd.

Condition SQ5 might seem a bit contrived, so we'll justify it here. Given a list of differences which satisfy SQ0-4, we still can't construct a grid, since the differences don't provide the cell values themselves. Entering a random value k into the first cell (prior to the first difference) then dictates rest of the grid, where we can begin to check the conditions. One might arbitrarily start by selecting diagonal d_i, locating the 0 in that diagonal, and finding the cell to the right of it to get a Referential Domino. Then, locating the cell it points to, for most choices of k this cell will not have the correct value (violates CN2). One could then increment k by 1, and notice some effects: the location of the 0 in the diagonal, and the cell to its right, decrease by 1 row (thus the cell they point to is the same location), and the value in the location the Referential Domino points to *increases* by 1. Thus, if we notice that for our choice of k this 0 Domino is *over* the correct value by 1, incrementing k by 1 will make it *under* the correct value by 1. Since these values are cyclic in $\mathbb{Z}_{n-1}$, if n is odd, then the amount we are off by must be even, else we will never reach the correct value.

For a given d_i and $d_{i'}$, let $m_i = [\sum_{j=0}^{i-1} d_j + \sum_{j=0}^{i'-1} d_j] \bmod n-1$. This value indicates which value of k in cell g(0,0) will make the diagonal d_i point to the correct value[1]. For any k, there is at most one m_i which produces it, thus to form a valid grid all m_i must be equal.

Below are two grids with the same difference list [3,2,4,6,5] with $n = 8$, $k = 2, 3$ respectively, to help illustrate:

G	0	1	2	3	4	5	6	7	—	0	1	2	3	4	5	6	7
0	**2**	7	3	6	1	**5**	4	0	—	**3**	7	4	0	2	**6**	5	1
1	5	3	7	4	0	2	6	1	—	6	4	7	5	1	3	0	2
2	0	6	4	7	5	1	3	2	—	1	**0**	**5**	7	6	2	4	3
3	4	1	**0**	**5**	7	6	2	3	—	5	2	1	6	7	0	3	4
4	3	5	2	1	6	7	0	4	—	4	6	3	2	0	7	1	5
5	1	4	6	3	2	0	7	5	—	2	5	0	4	3	1	7	6
6	7	2	5	0	4	3	1	6	—	7	3	6	1	5	4	2	0
7	6	0	1	2	3	4	5	7	—	0	1	2	3	4	5	6	7

For $k = 2$, g(0,5) should be 3 but is 5 (over by 2). For $k = 3$, g(0,5) should be 2 but is 6 (over by 4).

[1] If n is odd and m_i is even, there are two such k: $\frac{n-3-m_i}{2}$, and $\frac{2n-4-m_i}{2}$. If n is even and m_i is odd, $k = \frac{n-3-m_i}{2}$. If n and m_i are even, $k = \frac{2n-4-m_i}{2}$.

6 Self-referential Arrays

This analysis reveals a new, related problem: the Self-referential array. If Q0 is true, these appear to be effectively the same problem.

6.1 Forming an Array

Given a natural number $n > 3$, construct an array[2] of integers $b_i : 2 \leq i \leq n-2$ which satisfies the following conditions:

A1. b_i are unique, and $2 \leq b_i \leq n-2$.

A2. $b_i + b_{n-i} = n$

A3. $b_{b_i} = i$

A4. For all $k \geq 1, \sum_{j=i}^{i+k-1} b_j \not\equiv k \pmod{n-1}$

i.e. the sum of k consecutive elements in the array is not congruent to k mod $n-1$.

A5. m is constant for all i, where $m_i = \sum_{j=2}^{i-1} b_j + \sum_{j=2}^{b_i - 1} b_j \pmod{n-1}$.

Now let us consider the sequence A' where a'_i is the number of distinct valid arrays b_i.

6.2 Examples

The array $[3, 2, 4, 6, 5]$ is a valid array for $n = 8$, since:

A1. $2, 3, 4, 5, 6$ are unique and in the range $2 \leq i \leq 6$

A2. $3 + 5 = 2 + 6 = 4 + 4 = 8$

A3. $b_3 = 2$, $b_2 = 3$, $\therefore b_{b_2} = 2$, etc.

A4. Note that the case $k = 1$ is trivially true. The remainder of this verification is left as an exercise.

A5. Consider the case $i = 5$. The first sum is $3 + 2 + 4 = 9$, and the second is $3 + 2 + 4 + 6 = 15$. Thus, $m_5 \equiv (9 + 15) \equiv 3 \bmod 7$, which is the same for all i.

The array $[3, 2, 5, 4]$ is not a valid array for $n = 7$, due to A5: $m = 3$, m and n are both odd.

Also, $a'_8 = 2$, by computer search. The two arrays are $[3, 2, 4, 6, 5]$ and its reverse $[5, 6, 4, 2, 3]$.

[2] 2-indexed!

6.3 Miscellaneous

A computer search will further provide the following beginning to the sequence, with n from 4 to 32:

$$A' = [1, 1, 0, 0, 2, 1, 0, 2, 0, 2, 0, 0, 2, 4, 0, 4, 0, 0, 0, 4, 0, 2, 0, 2, 0, 6, 0, 4, 6]$$

Note that A is equal to A' at the even indices, and is double A' at the odd indices. For n which we can confirm with computer search, A' seems to have the same properties as A, such as only being non-zero for prime powers.